Phase Transitions and Quantum Measurements

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Abstract. In a quantum measurement, a coupling g between the system S and the apparatus A triggers the establishment of correlations, which provide statistical information about S. Robust registration requires A to be macroscopic, and a dynamical symmetry breaking of A governed by S allows the absence of any bias. Phase transitions are thus a paradigm for quantum measurement apparatuses, with the order parameter as pointer variable. The coupling g behaves as the source of symmetry breaking. The exact solution of a model where S is a single spin and A a magnetic dot (consisting of N interacting spins and a phonon thermal bath) exhibits the reduction of the state as a relaxation process of the off-diagonal elements of $\mathrm{S}+\mathrm{A}$, rapid due to the large size of N. The registration of the diagonal elements involves a slower relaxation from the initial paramagnetic state of A to either one of its ferromagnetic states. If g is too weak, the measurement fails due to a "Buridan's ass" effect. The probability distribution for the magnetization then develops not one but two narrow peaks at the ferromagnetic values. During its evolution it goes through wide shapes extending between these values.

QUANTUM MEASUREMENTS, A PROBLEM OF STATISTICAL MECHANICS

A quantum measurement has well known features which relate the initial state of the tested system S and the final state of the compound system $\mathrm{S}+\mathrm{A}$ including S and the apparatus A [1]. Von Neumann's reduction of the state postulates that an ideal measurement erases the off-diagonal blocks of the density matrix of $\mathrm{S}+\mathrm{A}$ in a basis where the measured observable is diagonal. Born's rule provides the probability of observing on the apparatus such or such value, and this value is correlated with the state of S after measurement. However, in order to understand how these features arise in an actual experiment, we need to analyze the dynamical process undergone by the coupled system $\mathrm{S}+\mathrm{A}$, to dig out the time scales involved, and to exhibit the specific properties of the quantum system A required so that it can be used as an apparatus.

At the initial time of the measurement, S and A are uncorrelated, with density operators $\hat{r}(0)$ and $\hat{\mathscr{R}}(0)$, respectively. The full density operator $\hat{\mathscr{D}}(t)$, initially equal to $\hat{\mathscr{D}}(0)=\hat{r}(0)\otimes\hat{\mathscr{R}}(0)$, evolves according to the Liouville–von Neumann equation. We wish to explain how it eventually reaches at the final time t_{f} the form

$$\hat{\mathscr{D}}(t_{\mathrm{f}})=\sum_i\left(\hat{\Pi}_i\hat{r}(0)\hat{\Pi}_i\right)\otimes\hat{\mathscr{R}}_i \tag{1}$$

which embeds the standard properties of ideal measurements recalled above. We denote by $\hat{\Pi}_i$ the projection operator over the eigenspace of the measured observable $\hat{A}$ labelled by its eigenvalue A_i, and by $\hat{\mathscr{R}}_i$ the set of possible final states of the apparatus A. Each one, $\hat{\mathscr{R}}_i$, is characterized by some value of a pointer variable of A, in one-to-one correspondence with the value A_i for S, and which can be observed or registered. The expression (1) means that, in a set of repeated experiments, a well defined outcome i occurs simultaneously for A (on which it can be observed) and for S, which is thereby prepared in the state $\hat{\Pi}_i\hat{r}(0)\hat{\Pi}_i/\,\mathrm{Tr}_{\mathrm{S}}\,\hat{\Pi}_i\hat{r}(0)$ through selection of the outcome i for A. Such events occur with a classical probability $\mathrm{Tr}_{\mathrm{S}}\,\hat{\Pi}_i\hat{r}(0)$.

The quantum system S is microscopic, and we wish its interaction with A to be sufficiently weak so that the final state involves no other correlation between S and A than the ones exhibited in eq. (1). The system A should therefore be able to switch from $\hat{\mathscr{R}}(0)$ to any one of the states $\hat{\mathscr{R}}_i$ under the effect of a very small perturbation. It is thus subject

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to a *bifurcation*. The state $\hat{\mathscr{R}}(0)$ of A should be metastable when the interaction with S is not yet turned on, and it should evolve towards any one of the states $\hat{\mathscr{R}}_i$ when triggered by this small interaction. Moreover, a robust and permanent registration excludes the possibility of transition from one state $\hat{\mathscr{R}}_i$ to another over a reasonable delay. Such states should thus be stable, and should have no overlap with one another, which imposes the apparatus A to be a macroscopic object. There should also exist some symmetry between the possible pointer variables, so that no bias is introduced by the apparatus. Otherwise the probability of observing the result i would not be $\mathrm{Tr_S}\,\hat{\Pi}_i\hat{r}(0)$ as stated by Born's rule, but would also depend on the lack of symmetry of A.

A natural means for achieving these requirements is to choose A as a macroscopic system which can undergo a phase transition with broken invariance. In some way or another, many actual apparatuses rely on such a transition (although bifurcations without broken symmetry are currently used). The pointer variable is then identified with the order parameter, a quantity that can be observed without perturbing the apparatus. The initial state $\hat{\mathscr{R}}(0)$ is chosen as the symmetric state which is metastable upon sudden cooling, and the states $\hat{\mathscr{R}}_i$ are the possible stable states with broken symmetry. Whereas in statistical mechanics the invariance is broken either spontaneously or under the effect of a scalar source, in a quantum measurement the breaking is triggered by coupling with S. Although this coupling should be weak, its effects on A are considerably amplified, due to the metastability of $\hat{\mathscr{R}}(0)$. On the other hand, we wish the measurement to perturb the microscopic system S as little as possible in spite of the macroscopic nature of A: we wish the diagonal blocks $\hat{\Pi}_i\hat{r}(0)\hat{\Pi}_i$ to be kept unchanged, while the laws of quantum mechanics should prevent the off-diagonal blocks of $\hat{r}(0)$ to subsist. Thus, although S behaves as a source for the phase transition, its quantum nature is essential for a full description of the dynamics.

From the large size of A and the small size of S we can anticipate that this dynamics of the coupled system $\mathrm{S}+\mathrm{A}$ during the measurement involves several time scales. The variables associated with S are expected to vary rapidly from the outset; during this stage A hardly changes. The establishment of correlations and the registration of the result in the form of a finite order parameter will require a much longer delay, which is required to let the macroscopic system A reach equilibrium.

A SOLVABLE MODEL

We have worked out the above ideas on an exactly solvable model, where S is a single spin $\frac{1}{2}$ represented by the Pauli operators $\hat{s}_x$, $\hat{s}_y$, $\hat{s}_z$, and where A simulates a magnetic dot containing N spins $\hat{\sigma}^{(n)}(n = 1, 2, \ldots N)$ which can undergo an Ising phase transition [2]. (For other models, see [3-8].) The order parameter is the average magnetization along z,

$$\hat{m} = \frac{1}{N}\sum_{n=1}^{N}\hat{\sigma}_z^{(n)}, \tag{2}$$

which takes at equilibrium the paramagnetic value $\langle\hat{m}\rangle = 0$ above the Curie temperature, or either one of the two ferromagnetic values $\langle\hat{m}\rangle = \pm m_{\mathrm{F}}$ below. In both cases the statistical fluctuation is small as $1/\sqrt{N}$. Thermal equilibrium of $\mathrm{A} = \mathrm{M}+\mathrm{B}$ is ensured by a weak interaction of these magnetic degrees of freedom M with a thermal bath B, represented by a model describing the phonons in the dot. The full Hamiltonian of $\mathrm{S}+\mathrm{A}$ has therefore the form $\hat{H} = \hat{H}_{\mathrm{SA}} + \hat{H}_{\mathrm{A}}$, where

$$\hat{H}_{\mathrm{SA}} = -g\hat{s}_z\sum_{n=1}^{N}\hat{\sigma}_z^{(n)} = -Ng\hat{s}_z\hat{m} \tag{3}$$

accounts for the interaction between the z-component of the spin $\hat{s}$ and the apparatus, and where $\hat{H}_{\mathrm{A}} = \hat{H}_{\mathrm{M}} + \hat{H}_{\mathrm{MB}} + \hat{H}_{\mathrm{B}}$ includes the Ising interaction

$$\hat{H}_{\mathrm{M}} = -\frac{J}{2N}\sum_{n,n'}\hat{\sigma}_z^{(n)}\hat{\sigma}_z^{(n')} = -\frac{NJ}{2}\hat{m}^2 \tag{4}$$

between the spins of the apparatus (which all interact pairwise, unlike in our previous studies, where quartets interact), the coupling

$$\hat{H}_{\mathrm{MB}} = \sqrt{\gamma}\sum_{n=1}^{N}\sum_{a=x,y,z}\hat{\sigma}_a^{(n)}\hat{B}_a^{(n)} \tag{5}$$

of these spins with the bath through phonon operators $\hat{B}_a^{(n)}$, and the phonon Hamiltonian $\hat{H}_{\mathrm{B}}$. The initial state $\hat{\mathscr{R}}(0) = \hat{R}_{\mathrm{M}}(0)\otimes\hat{R}_{\mathrm{B}}(0)$ of A factorizes into contributions of the magnet and of the bath, where $\hat{R}_{\mathrm{M}}(0) = \hat{I}/2^N$

represents a completely disordered paramagnetic state, prepared by bringing M at a high temperature before the measurement, and where $\hat{R}_{\mathrm{B}}(0) \propto e^{-\hat{H}_{\mathrm{B}}/T}$ describes equilibrium of the phonons at the temperature T. This temperature lies below the Curie temperature ($T < J$) so that $\hat{\mathscr{R}}(0)$ is metastable: it may transit towards the stable ferromagnetic states, in a rather large time if the magnet-bath coupling γ is weak. The bath Hamiltonian $\hat{H}_{\mathrm{B}}$ will enter the problem only through the autocorrelation function $\left\langle \hat{B}_a^{(n)}(t)\,\hat{B}_{a'}^{(n')}(t') \right\rangle$, which under rather general conditions has the form

$$\mathrm{Tr}\,\hat{R}_{\mathrm{B}}(0)\,\hat{B}_a^{(n)}(t)\,\hat{B}_{a'}^{(n')}(t') = \delta_{nn'}\delta_{aa'}K\left(t-t'\right) \ , \tag{6}$$

$$\tilde{K}(\omega) = \int_{-\infty}^{+\infty} dt e^{-i\omega t} K(t) = \frac{\hbar^2\omega}{4}\frac{e^{-|\omega|/\Gamma}}{e^{\hbar\omega/T}-1} \ . \tag{7}$$

We have denoted the Debye cutoff as Γ.

Since γ is weak, the bath can be eliminated by means of a second-order treatment of $\hat{H}_{\mathrm{MB}}$. From the Liouville–von Neumann equation for $\hat{\mathscr{D}}$, we thus find, for the partial trace $\hat{D} = \mathrm{Tr}_{\mathrm{B}}\,\hat{\mathscr{D}}$ which describes the joint evolution of S and M (in the presence of the bath), the equation

$$\frac{d\hat{D}}{dt} = \frac{1}{i\hbar}\left[\hat{H}_{\mathrm{SA}} + \hat{H}_{\mathrm{M}}, \hat{D}\right] + \frac{\gamma}{\hbar^2}\sum_{n=1}^{N}\sum_{a=x,y,z}\int_0^t dt' \left[\hat{\sigma}_a^{(n)},\, \hat{D}(t)\,K\left(-t'\right)\hat{\sigma}_a^{(n)}\left(t'\right) - K\left(t'\right)\hat{\sigma}_a^{(n)}\left(t'\right)\hat{D}(t)\right] \ . \tag{8}$$

We have introduced, in the interaction representation,

$$\hat{\sigma}_a^{(n)}(t) \equiv \hat{U}(t)\,\hat{\sigma}_a^{(n)}\hat{U}^{\dagger}(t) \ , \tag{9}$$

$$\hat{U}(t) = \exp\left[\left(\hat{H}_{\mathrm{SA}} + \hat{H}_{\mathrm{M}}\right)t/i\hbar\right] \ . \tag{10}$$

DISAPPEARANCE OF SCHRÖDINGER CATS

During an ideal quantum measurement, the quantity to be measured should not change. This is expressed here by the commutation of $\hat{H}$ with $\hat{s}_z$, or equivalently with the projection operators $\hat{\Pi}_i$ on the eigenstates $i = \uparrow$ or $\downarrow$ of $\hat{s}_z$. Hence, eq. (8) can be decomposed into four equations governing the blocks $\hat{D}_{ij} \equiv \hat{\Pi}_i\hat{D}\hat{\Pi}_j\,(i, j = \uparrow \text{ or } \downarrow)$. Each $\hat{D}_{ij}$ is an operator in the space of M.

This decoupling allows us to treat separately the off-diagonal and diagonal blocks. The evolution of $\hat{D}_{\uparrow\downarrow} = \hat{D}_{\downarrow\uparrow}^{\dagger}$ governed by (8) has been studied elsewhere [2, 9]. We briefly recall the results here. During a very brief reduction time $\tau_{\mathrm{red}} = \hbar/\sqrt{2N}g$, all elements of $\hat{D}_{\uparrow\downarrow}$, which describe correlations between the components $\hat{s}_x$ or $\hat{s}_y$ of the spin S and the various spins $\hat{\sigma}^{(n)}$ of A, decrease down to zero. The initial order exhibited by the non-vanishing value of $\langle\hat{s}_x\rangle$ or $\left\langle\hat{s}_y\right\rangle$ is scattered into a very large number of small correlations between $\hat{s}_x$ or $\hat{s}_y$ and the z-component of the many spins of A.

This relaxation process is governed by the interaction term $\hat{H}_{\mathrm{SA}}$. If the Hamiltonian if $\mathrm{S}+\mathrm{A}$ did reduce to this term, the lost order would surge back in $\hat{D}_{\uparrow\downarrow}$ at the time $\pi\hbar/(2g)$, producing detectable effects through a recurrence process. Such recurrences are hindered, and irreversibility of the collapse is ensured, owing to the presence of the spin-bath coupling term $\hat{H}_{\mathrm{MB}}$, provided $\gamma \gg g^2/N\hbar^2\Gamma^2$. The initial order is then lost once and for all into the bath degrees of freedom. In the language of NMR, this irretrievable loss of phase coherence is similar to a spin-lattice relaxation process. However, an alternative mechanism involving no phonon bath can also suppress the recurrences, for an apparatus A consisting only of the magnet M. Indeed, all the matrix elements of $\hat{D}_{\uparrow\downarrow}$ remain negligible at all times after τ_{red} if the system-magnet interaction $\hat{H}_{\mathrm{SA}}$ involves slightly different coupling constants g_n between $\hat{s}_z$ and the apparatus spins $\hat{\sigma}_z^{(n)}$, such that their relative fluctuation satisfies $\delta g/g \gg 1/\sqrt{N}$. This second mechanism is comparable to the relaxation in NMR due to inhomogeneity of the external Larmor field. As in Hahn's spin echoes, one can imagine to retrieve here the initial order associated with $\hat{D}_{\uparrow\downarrow}$ by means of an adequate setup of pulses.

Altogether, the von Neumann collapse of the state, which can be regarded as the disappearance of "Schrödinger cats" in a measurement, is explained as a relaxation process, which is rapid due to the large size of N. An important role is also played by the mixed nature of the initial state of our apparatus. In most theoretical discussions of the measurement

process a pure initial state is assumed. This is not realistic, however, since it is impossible to control the macroscopic number of degrees of freedom of an apparatus. The relaxation related to the collapse should not be confused with standard decoherence. It does not necessarily require the thermal bath, it depends on the measured observable, and it takes place over a time $\tau_{\rm red}$ which involves g, not T.

REGISTRATION BY THE APPARATUS

We now focus on the evolution of the diagonal blocks $\hat{D}_{\uparrow\uparrow}$ and $\hat{D}_{\downarrow\downarrow}$, which describe the expectation values of $\hat{s}_z$ and of the spin operators of M, their fluctuations and all their correlations. This evolution should end up in a final state of the form (1), expressing the registration of the measurement: a complete correlation established between the final value ± 1 of $\hat{s}_z$ and the final ferromagnetic equilibrium state, characterized by the sign of the order parameter $\langle \hat{m} \rangle = \pm m_{\rm F}$. The diagonal block $\hat{D}_{\uparrow\uparrow}$ is associated with the occurrence of $s_z = +1$. We have to prove that in this block the probability distribution for $\hat{m}$ tends for large times to become sharply peaked around $+m_{\rm F}$, without any contribution from the region of $-m_{\rm F}$. In the equation of motion for $\hat{D}_{\uparrow\uparrow}$, obtained from (8), the operator $\hat{H}_{\rm SA} + \hat{H}_{\rm M}$ depends only on the eigenvalue of $\hat{s}_z$ in this sector, equal here to $+1$, and on the observable $\hat{m}$, the successive eigenvalues of which are separated by a distance $2/N$. This introduces through (3), (4) and (9), (10) the operators

$$\hbar\hat{\Omega}_{\pm} = -Ng\left[\left(\hat{m} \pm \frac{2}{N}\right) - \hat{m}\right] - \frac{NJ}{2}\left[\left(\hat{m} \pm \frac{2}{N}\right)^2 - \hat{m}^2\right] = \mp 2\left(g + J\hat{m}\right) - 2J/N \equiv \mp 2h\left(\hat{m}\right) - 2J/N \,, \tag{11}$$

where $h\left(\hat{m}\right) = g + J\hat{m}$ behaves as an operator-valued self-consistent field. We also introduce the function

$$\tilde{K}_t\left(\omega\right) \equiv \int_{-t}^{+t} ds e^{-i\omega s} K\left(s\right) = \int_{-\infty}^{+\infty} d\omega' \tilde{K}\left(\omega'\right) \frac{\sin\left(\omega' - \omega\right) t}{\pi\left(\omega' - \omega\right)} \,, \tag{12}$$

in which ω can be replaced by the operators $\hat{\Omega}_{\pm}$, and thus find the reduced equation of motion

$$\frac{d\hat{D}_{\uparrow\uparrow}}{dt} = \frac{2\gamma}{\hbar^2}\sum_{n=1}^{N}\left[\hat{\sigma}_-^{(n)},\, \hat{D}_{\uparrow\uparrow}\tilde{K}_t\left(\hat{\Omega}_-\right)\hat{\sigma}_+^{(n)} - \hat{\sigma}_+^{(n)}\tilde{K}_t\left(\hat{\Omega}_+\right)\hat{D}_{\uparrow\uparrow}\right] \,. \tag{13}$$

This evolution conserves the trace $\mathrm{Tr}_{\rm M}\,\hat{D}_{\uparrow\uparrow} = \mathrm{Tr}_{\rm S}\,\hat{\Pi}_{\uparrow}\hat{r}\left(0\right) = r_{\uparrow\uparrow}(0)$, a normalization consistent with Born's rule. Moreover the operator $\hat{D}_{\uparrow\uparrow}$ in the space of M turns out to be simply a function of $\hat{m}$ at each time, $\hat{D}_{\uparrow\uparrow}\left(t\right) = \Delta_{\uparrow\uparrow}\left(\hat{m}, t\right)$, since this property holds at the initial time and is preserved by the motion (13). In fact, the knowledge of $\Delta_{\uparrow\uparrow}\left(\hat{m}, t\right)$ is equivalent to that of the conditional probability $P_{\rm d}\left(m, t\right)$ for $\hat{m}$ to take the discrete values $m = -1$, $-1 + 2/N$, ..., $1 - 2/N$, 1 if s_z equals $+1$, which is expressed by

$$P_{\rm d}\left(m, t\right) = \frac{N!}{\left[\frac{1}{2}N\left(1+m\right)\right]!\left[\frac{1}{2}N\left(1-m\right)\right]!}\frac{\Delta_{\uparrow\uparrow}\left(m, t\right)}{\mathrm{Tr}\hat{D}_{\uparrow\uparrow}} \,. \tag{14}$$

From (13) we find the equation of motion for $P_{\rm d}\left(m, t\right)$,

$$\begin{aligned}\frac{\partial P_{\rm d}\left(m, t\right)}{\partial t} = \frac{\gamma N}{\hbar^2}\Bigg[&\tilde{K}_t\left(-\Omega_+\right)\left(1 + m + \frac{2}{N}\right)P_{\rm d}\left(m + \frac{2}{N}, t\right) - \tilde{K}_t\left(\Omega_+\right)\left(1 - m\right)P_{\rm d}\left(m, t\right) \\ &+ \tilde{K}_t\left(-\Omega_-\right)\left(1 - m + \frac{2}{N}\right)P_{\rm d}\left(m - \frac{2}{N}, t\right) - \tilde{K}_t\left(\Omega_-\right)\left(1 + m\right)P_{\rm d}\left(m, t\right)\Bigg] \,,\end{aligned} \tag{15}$$

where $\Omega_{\pm}$ are functions of m defined by eq.(11). For shorthand we have denoted by $P_{\rm d}\left(m, t\right)$ instead of $P_{\uparrow}\left(m, t\right)$ the conditional probability of m associated with the value $s_z = +1$ of the measured observable of S, while the subscript 'd' indicates that m is discrete here. We can introduce likewise for the sector $\downarrow\downarrow$ the conditional probability $P_{\downarrow}\left(m, t\right)$ associated with $s_z = -1$, which is obtained by merely changing g into $-g$ in the expression (11) of $\Omega_{\pm}$.

The registration process of the measurement is therefore just the same, in the sector $\uparrow\uparrow$, as the relaxation of the Ising model towards equilibrium under the influence of a field $+g$ and of the phonon bath, a problem that we now study. The dynamical equation (15) has been derived from the Hamiltonian $\hat{H}_{\rm A} - Ng\hat{m}$ without any other approximation than

a weak magnet-bath coupling $\gamma \ll 1$. The bath occurs through $\tilde{K}_t$ defined by (7), (12), while the Hamiltonian of M (including the field $+g$) occurs through the energy shifts $\hbar\Omega_\pm$ given by (11).

For sufficiently small γ the evolution is slow and its time scale is large compared to $\hbar/T$, so that $\tilde{K}_t(\omega)$ can be replaced in (15) by $\tilde{K}(\omega)$. In such a short-memory approximation, we have $\tilde{K}(-\omega) = \tilde{K}(\omega)e^{\hbar\omega/T}$, and the equation (15) for $P_{\mathrm{d}}(m,t)$ can be identified with a balance equation, where the probability of each spin flip induced by the coupling with the bath is given by Fermi's golden rule, and which might have been written directly on phenomenological grounds. In terms of the entropy S and the energy U of M associated with the density operator $\hat{D}_{\uparrow\uparrow}$, eq. (15) satisfies in this regime an H-theorem

$$\frac{d}{dt}\left(S - \frac{U}{T}\right) = \frac{\gamma N}{\hbar^2}\sum_m \tilde{K}(\Omega_-)\left[e^{\hbar\Omega_-/T}\left(1 - m + \frac{2}{N}\right)P_{\mathrm{d}}\left(m - \frac{2}{N}\right) - (1+m)P_{\mathrm{d}}(m)\right] \tag{16}$$
$$\ln\frac{e^{\hbar\Omega_-/T}\left(1 - m + \frac{2}{N}\right)P_{\mathrm{d}}\left(m - \frac{2}{N}\right)}{(1+m)P_{\mathrm{d}}(m)} \geq 0 \,,$$

which implies that $\hat{D}_{\uparrow\uparrow}$ tends within normalization to the equilibrium distribution $\exp N\left(g\hat{m} + \frac{1}{2}J\hat{m}^2\right)/T$, with possible invariance breaking for small g.

DYNAMICS OF THE PHASE TRANSITION

Our purpose is to study the dynamics of this relaxation, which for $T < J$ involves a bifurcation from the vicinity of $m = 0$ towards that of $\pm m_{\mathrm{F}}$. In the large N limit, we may treat m as a continuous variable, $P(m,t) = (N/2)P_{\mathrm{d}}(m,t)$ being now normalized as $\int_{-1}^{+1} dm\, P(m,t) = 1$. From (7), (11) and (15) we get for $t \gg \hbar/T$, keeping the terms of order 1 and $1/N$,

$$\frac{\partial P(m,t)}{\partial t} = \frac{\partial}{\partial m}\left[-v(m)P(m,t) + \frac{1}{N}w(m)\frac{\partial P(m,t)}{\partial m}\right] , \tag{17}$$

where, with now $h(m) = g + Jm$ just involving the c-number m,

$$v(m) \equiv \frac{\gamma h(m)}{\hbar}\left(1 - m\coth\frac{h(m)}{T} + \frac{1}{N}\right) , \tag{18}$$

$$w(m) \equiv \frac{\gamma h(m)}{\hbar}\left(\coth\frac{h(m)}{T} - m\right) . \tag{19}$$

This type of equation has been extensively studied [10, 11]. We analyse below its solution in the present context. The term in $1/N$ of (17) is negligible for smooth probabilities, but not for sharply peaked functions $P(m)$, which occur at least in the initial state and in the final equilibrium ferromagnetic states. In fact, this term is dominant in the vicinity of the points where $m = \tanh h/T$ for which $v(m)$ vanishes. In particular, as implied by the inequality (16), $P(m,t)$ tends for large times to an equilibrium shape characterized by the vanishing of the square bracket in (17). This condition entails

$$P(m,\infty) \propto \exp\left[-\frac{N}{2}\left(\frac{m - m_{\mathrm{F}}}{\delta_{\mathrm{F}}}\right)^2\right] , \tag{20}$$

where m_{F} and δ_{F} are given by

$$m_{\mathrm{F}}\left(1 - \frac{1}{N}\right) = \tanh\frac{g + Jm_{\mathrm{F}}}{T} \quad , \quad m_{\mathrm{F}} \gtrless 0 \,, \tag{21}$$

$$\frac{1}{\delta_{\mathrm{F}}^2} = \frac{1}{w}\frac{dv}{dm}\bigg|_{m_{\mathrm{F}}} = \frac{1}{1 - m_{\mathrm{F}}^2} - \frac{J}{T} > 0 \,. \tag{22}$$

As readily checked, we recover dynamically the expected canonical distribution, including corrections for finite but large N. However, for $g \ll Jm_{\mathrm{F}}^3$, eq. (21) has two solutions such that (22) is positive, with nearly opposite values $\pm m_{\mathrm{F}}$. Thus the asymptotic limit of $P(m,t)$ for large t is not a priori known: it is a linear combination of the two ferromagnetic distributions (20) peaked around $+m_{\mathrm{F}}$ and $-m_{\mathrm{F}}$. The present dynamical approach is necessary to explain how the

evolution towards equilibrium produces the breaking of invariance. In fact, the weights $\mathscr{P}_+$ and $\mathscr{P}_-$ of the two terms $+m_{\mathrm{F}}$ and $-m_{\mathrm{F}}$ can only be found by solving the dynamical equation (17), which will allow us to express them in terms of the initial condition $P(m,0)$. In the measurement problem, it is essential that in the $\uparrow\uparrow$ sector, for $g>0$, the probability $P(m,t)$ concentrates for large times only around the positive ferromagnetic value $+m_{\mathrm{F}}$; a finite weight $\mathscr{P}_-$ for the peak $-m_{\mathrm{F}}$ would mean a wrong indication of the apparatus for the system in the state $s_z=+1$. Moreover, we need the lifetime of the initial paramagnetic state to be sufficiently large so that the interaction between S and A can be turned on while M still lies in the metastable paramagnetic state. In the theory of phase transitions, we wish to find through which intermediate shapes $P(m,t)$ passes when the magnet M relaxes from the initial state to either one of the ferromagnetic equilibrium states, and to determine the probabilities of both occurrences.

Our equation (17), which governs the time-dependence of the order parameter, treated as a random variable, has the same form as the Fokker–Planck equation for a Brownian particle in one dimension submitted to an external influence. Its first term represents a deterministic drift, its second term a diffusion process which tends to widen the distribution $P(m,t)$. Let us first drop this diffusion term. Eq. (17) has then elementary solutions of the form $\delta(m-m(\mu,t))$, where $m(\mu,t)$ is the trajectory of a particle with initial position $m=\mu$ and (position-dependent) velocity $dm/dt=v(m)$. (For very short times, which are not relevant below, we find from (12), (15) that v also depends on time, as $v\sim-\pi^{-1}\gamma\Gamma^2 tm$, the sign of which implies that the fixed point $m=0$ is initially stable.) The functions $m(\mu,t)$ is obtained by inverting the equation

$$t=\int_\mu^m \frac{dm'}{v(m')}=\frac{\hbar}{\gamma}\int_\mu^m \frac{dm'}{h(m')\left[1-m'\coth h(m')/T\right]}\ . \tag{23}$$

This motion of m has three fixed points, the zeroes of $v(m)$. The closest to the origin, which for $g\ll Jm_{\mathrm{F}}^3$ lies at

$$m_{\mathrm{P}}=-\frac{g}{J-T}\ , \tag{24}$$

is repulsive: if μ differs slightly from m_{P}, the point $m(\mu,t)$ moves astray. The other two fixed points are attractors associated with the ferromagnetic phases near $+m_{\mathrm{F}}$ and $-m_{\mathrm{F}}$. Thus, if the diffusion term is discarded, the initial probability $P(m,0)$ is split into two parts $m>m_{\mathrm{P}}$ and $m<m_{\mathrm{P}}$, which will eventually result into sharp peaks located at $+m_{\mathrm{F}}$ and $-m_{\mathrm{F}}$, respectively. In the evolution of $P(m,t)$, the weight $P(m,t)\,dm$ is conserved along the motion (23). Hence, denoting by $\mu(m,t)$ the inverse mapping of $m(\mu,t)$, the solution of eq. (17) without the diffusion term is [11]

$$P(m,t)=P[\mu(m,t),0]\,\frac{v[\mu(m,t)]}{v(m)}\ , \tag{25}$$

since $dm/d\mu=v(m)/v(\mu)$ for given t. We denoted $v(\mu(m,t))$ as $v[\mu(m,t)]$ and $P(\mu(m,t),0)$ as $P[\mu(m,t),0]$.

However, as we already noted, diffusion is essential in the first stage of the motion, when $P(m,0)$ is concentrated around small values of m where $v(m)$ is small. In this region, it is possible to solve the full equation (17) by expanding $v(m)\approx(\gamma/\hbar)[g+(J-T)m]$ and $w(m)\approx\gamma T/\hbar$ in powers of m, leading to

$$\frac{\hbar}{\gamma}\frac{\partial P}{\partial t}=-\frac{\partial}{\partial m}\left\{[g+(J-T)\,m]\,P\right\}+\frac{T}{N}\frac{\partial^2 P}{\partial m^2}\ . \tag{26}$$

We can then find the explicit solution of (23), and invert the mapping $m(\mu,t)$,

$$m(\mu,t)=\mu\,e^{t/\theta}+\frac{g}{J-T}\left(e^{t/\theta}-1\right),\qquad \mu(m,t)=me^{-t/\theta}-\frac{g}{J-T}\left(1-e^{-t/\theta}\right)\ , \tag{27}$$

where we introduced the time-scale

$$\theta\equiv\frac{\hbar}{\gamma(J-T)}\ . \tag{28}$$

The solution of (26), found by means of a Fourier transform on m, takes after some calculations the form

$$P(m,t)=\sqrt{\frac{N}{2\pi C}}\int d\xi\, e^{-N\xi^2/2C}P[\mu(m,t)+\xi,0]\,\frac{v[\mu(m,t)]}{v(m)}\ , \tag{29}$$

where

$$C\equiv\left(1-e^{-2t/\theta}\right)\frac{T}{J-T}. \tag{30}$$

The result (29) encompasses the drift induced by the first term of (26) and the diffusion induced by its second term. It turns out that the cumulated effect of diffusion is equivalent to a blurring of the initial value from which m is issued in the deterministic motion (27), over a width $\sqrt{C}$ which increases with t. We note however that, for $e^{t/\theta} \gg 1$ the effect of diffusion remains constant, as C tends then to $C_\infty = T/(J-T)$.

Since, according to (27), θ is also the characteristic time over which $m(\mu,t)$ exponentially diverges from its initial value μ, we can match the effect (29) of diffusion with the effect (25) of drift, even in regions where m is no longer small, by writing the solution of (17) in the form (29) where $\mu(m,t)$ is now obtained from (23) rather than from (27). This result holds, up to the very large times when the expression (29) gets concentrated near $\pm m_\mathrm{F}$. In the later stage of the evolution, that we do not need to consider, the diffusion term becomes again effective, and it determines the shape and width of the final probability according to (20)-(22). In order to discuss the behavior of (29) we note that when $J-T \ll J$, $m_\mathrm{F}^2 \sim 3(J-T)/J$, one has $m_\mathrm{P} \sim -g/(J-T) \ll m_\mathrm{F}$, while $v(m)$ has the form

$$v(m) \approx \frac{(m-m_\mathrm{P})}{\theta} \frac{m_\mathrm{F}^2 - m^2}{m_\mathrm{F}^2} . \tag{31}$$

More generally, we note that (31) has for $g \ll J m_\mathrm{F}^3$ but arbitrary $T < J$ the same features as (18): same zeroes, same behavior for small m. For this qualitatively good model of $v(m)$ we can integrate explicitly (23) for $|m_\mathrm{P}| \ll m_\mathrm{F}$ as

$$e^{t/\theta} = \frac{m-m_\mathrm{P}}{\mu-m_\mathrm{P}} \sqrt{\frac{m_\mathrm{F}^2-\mu^2}{m_\mathrm{F}^2-m^2}} , \tag{32}$$

or equivalently

$$m(\mu,t) = \frac{\left[\mu e^{t/\theta} - m_\mathrm{P}\left(e^{t/\theta}-1\right)\right] m_\mathrm{F}}{\sqrt{m_\mathrm{F}^2 + (\mu-m_\mathrm{P})^2\left(e^{2t/\theta}-1\right)}} , \tag{33}$$

$$\mu(m,t) = m_\mathrm{P} + \frac{(m-m_\mathrm{P})\, e^{-t/\theta} m_\mathrm{F}}{\sqrt{m_\mathrm{F}^2 - m^2\left(1-e^{-2t/\theta}\right)}} . \tag{34}$$

We take as initial state the narrow distribution

$$P(m,0) = \sqrt{\frac{N}{2\pi\delta_0^2}} \exp\left[-\frac{N}{2}\left(\frac{m-m_0}{\delta_0}\right)^2\right] , \tag{35}$$

where m_0 characterizes a possible deviation from the paramagnetic state (for which $m_0 = 0$). The width δ_0, determined by the initial equilibrium of M at a temperature T_0 higher than J, is given by $\delta_0^2 = T_0/(T_0 - J)$. When quenching from $T_0 = \infty$, as is done in the figures, one has $m_0 = 0$ and $\delta_0 = 1$. Use of (32) in (29) and integration yield

$$P(m,t) = \sqrt{\frac{N}{2\pi\left(C+\delta_0^2\right)}} \exp\left[-\frac{N}{2}\frac{(\mu-m_0)^2}{C+\delta_0^2}\right] \frac{v[\mu(m,t)]}{v(m)} , \tag{36}$$

where C is given by (30), μ is the function of m and t given by (34), and $\partial\mu/\partial m = v(\mu)/v(m)$.

The expression (36) holds at all times, except in the very last stage of the evolution, when $P(m,t)$, already concentrated near $+m_\mathrm{F}$ and $-m_\mathrm{F}$, switches into the shape (20). We can use it, however, to evaluate the probability $\mathscr{P}_+$ or $\mathscr{P}_-$ that the initial state (35) ends up at one or the other ferromagnetic states $+m_\mathrm{F}$ or $-m_\mathrm{F}$. Indeed, in the mapping $m(\mu,t)$, the points μ associated with the vicinity of the final point $m = +m_\mathrm{F}$ are those for which $\mu > m_\mathrm{P}$. Taking thus μ instead of m as an integration variable in (36) and letting $t \to \infty$, we find

$$\mathscr{P}_- = \frac{1}{2}\,\mathrm{erfc}\left[\sqrt{\frac{N}{2}}\frac{b}{\delta}\right] , \tag{37}$$

where

$$b \equiv -m_\mathrm{P} + m_0 = \frac{g}{J-T} + m_0 \tag{38}$$

measures the bias due to the external field g and to the shift m_0 in the initial distribution, where

$$\delta^2 \equiv C_\infty + \delta_0^2 = \frac{T}{J-T} + \delta_0^2 \tag{39}$$

accounts for the modification of the initial width $\delta_0/\sqrt{N}$ due to diffusion, and where

$$\mathrm{erfc}\,(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \ . \tag{40}$$

UNIQUE FINAL STATE

We can therefore distinguish two regimes which display several qualitatively different features. In the first case, the argument of erfc in (37) is large, so that the magnet M nearly certainly reaches the ferromagnetic state $+m_{\mathrm{F}}$: the bifurcation is ineffective. This occurs, for an unbiased initial state ($m_0 = 0$), when

$$g \gg \frac{(J-T)\,\delta}{\sqrt{N}} \ ; \tag{41}$$

this also occurs for $g = 0$ when the shift m_0 of the initial state with respect to paramagnetism satisfies

$$m_0 \gg \frac{\delta}{\sqrt{N}} \ . \tag{42}$$

In this situation, the duration of the relaxation process is of order $\theta = \hbar/\gamma(J-T)$. More precisely, if we consider that ferromagnetism is reached when the distribution $P\,(m,t)$ is peaked near $m = 0.95\ m_{\mathrm{F}}$, this time, the registration time for a measurement, is found from (32) to be

$$\tau_{\mathrm{reg}} = \frac{\hbar}{\gamma(J-T)} \ln \frac{3m_{\mathrm{F}}}{b} \ . \tag{43}$$

Since the range $t < \tau_{\mathrm{reg}}$ does not involve N, it is seen from (34) that N occurs in the expression (36) of $P\,(m,t)$ only as a factor in the exponent. Hence $P\,(m,t)$ displays at all times a single peak, see Fig. 1, that is narrow as $1/\sqrt{N}$ and located at the point m where $\mu\,(m,t)$ equals m_0. The position of this peak is thus given by (33) in which μ is replaced by m_0. Its width is seen from (36) to equal $\delta\,(t)\,/\sqrt{N}$, where

$$\delta\,(t) = \sqrt{C + \delta_0^2}\ \frac{v\,(m)}{v\,(\mu)} \ . \tag{44}$$

When $m\,(m_0,t)$ is still close to the origin, diffusion lets C increase up to $C_\infty = T/\,(J-T)$. The widening of the distribution $P\,(m,t)$ is then governed by the dispersion of the speed $v\,(m)$ of the drift motion [10]: the head of the distribution, with larger m, proceeds faster than its tail. The maximum of $\delta\,(t)$ is thus attained when the peak of $P\,(m,t)$ is located at the value of m which lets $v\,(m)$ be maximum, that is, $m = m_{\mathrm{F}}/\sqrt{3}$, reached for

$$t = \frac{\hbar}{\gamma(J-T)} \ln \frac{m_{\mathrm{F}}}{\sqrt{2}\ b} \ . \tag{45}$$

We have then

$$\delta_{\max} = \frac{2m_{\mathrm{F}}\delta}{3\sqrt{3}\ b} \ . \tag{46}$$

Later on, $\delta\,(t)$ decreases, again due to the gradient in the drift velocity, which is now negative. This effect is, in the last stage, counterbalanced by diffusion which prevents $\delta\,(t)$ from decreasing below the equilibrium value δ_{F} given by (22).

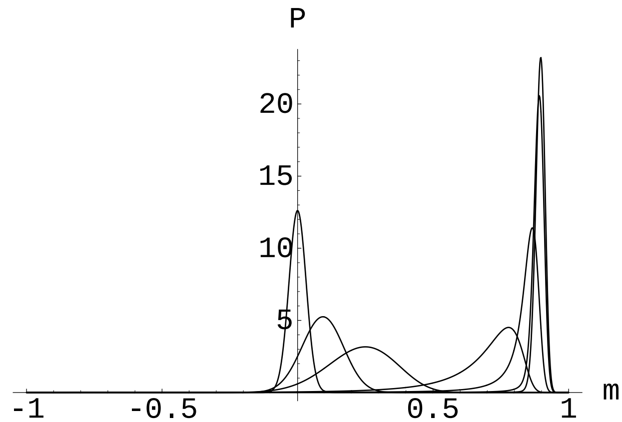

FIGURE 1. Transition from the initial paramagnetic state to the ferromagnetic state with magnetization up, driven by the coupling g. We have solved eq. (15) in the short-memory approximation, valid for $\gamma \ll 1$, where $\tilde{K}_t(\omega)$ is replaced by (7), the Debye cutoff being irrelevant. The parameters were chosen as $N = 1000$, $T = 0.65J$, $g = 0.05J$, small but satisfying the condition (41). The curves describe the probability distribution $P(m,t)$ at the times (from left to right) $t/\theta = 0$, 0.5, 1, 2.25, 3, 4 and 5; the latter curve is near the infinite time limit. As initial distribution we took (35) with $m_0 = 0$ and $\delta_0 = 1$, corresponding to the paramagnet $T_0 = \infty$. The center of the peak moves from $m = 0$ to $m_F = 0.89707$ according to (23); its width first increases up to (46) then decreases down to (22).

ACTIVE BIFURCATION

We now consider the second regime, for which the bifurcation is active in the sense that neither $\mathscr{P}_+$ nor $\mathscr{P}_-$ vanish. Such a possibility for a macroscopic system to transit towards several (here two) possible final states is known as spinodal decomposition. This situation is a nuisance for the measurement problem, since it corresponds to a failure in the establishment of correlations between the indication of A and the residual state of S. For the phase transition problem, it corresponds to dynamics for which the external field g and the initial shift m_0 are too small to fully determine the phase towards which the system A will relax. Since at large times $P(m,t)$ must have two sharp peaks, we know that some samples will reach $+m_{\rm F}$, other ones $-m_{\rm F}$, but we are interested here in the history of these samples. A natural question is the following: do the samples that will end up at $+m_{\rm F}$ behave as above, with a well-defined trajectory $m(t)$ within fluctuations of order $1/\sqrt{N}$? More precisely, how does the distribution $P(m,t)$ evolve before it displays its final two-peaked equilibrium shape?

To answer this question, which arises when the conditions (41), (42) are violated, we assume that the bias due to both the external field and the initial condition is small for large N as

$$b \equiv -m_{\rm P} + m_0 = \frac{g}{J-T} + m_0 = \sqrt{\frac{2}{N}}\,\lambda\,\delta\;, \tag{47}$$

where δ is defined by (39) and where λ is either finite or small. The probability distribution, given by (36), begins to widen under the effect of diffusion as

$$P(m,t) = \sqrt{\frac{N}{2\pi\left(C+\delta_0^2\right)}}\exp\left[-\frac{N}{2}\frac{\left[m-m_0+m_{\rm P}\left(1-e^{-t/\theta}\right)\right]^2}{C+\delta_0^2}\right]\;, \tag{48}$$

where C increases according to (30). Even in case the initial distribution (35) extends only over values larger than $m_{\rm P}$, that is, for $\delta_0 \ll b = m_0 - m_{\rm P}$, (48) develops a tail which may extend to the region $m < m_{\rm P}$, and which will give rise to the non-vanishing probability (37). This initial widening is relayed by the widening due to the gradient of the drift velocities, which will have here dramatic consequences. Indeed, whereas in the first regime the whole relaxation process was achieved after a delay (43) independent of N, the probability is now not yet concentrated near $\pm m_{\rm F}$ for times such that $e^{t/\theta}$ is of order $\sqrt{N}$, see Fig. 2. In such a range of times, the distribution (36) takes the form

$$P(m,t) = \frac{\alpha m_{\rm F}^2}{\sqrt{\pi}\,\left(m_{\rm F}^2-m^2\right)^{3/2}}\exp\left[-\left(\frac{\alpha m}{\sqrt{m_{\rm F}^2-m^2}}-\lambda\right)^2\right]\;, \tag{49}$$

where we introduced Suzuki's scaling variable [11]

$$\alpha \equiv \sqrt{\frac{N}{2}}\, e^{-t/\theta}\, \frac{m_{\mathrm{F}}}{\delta} \; . \tag{50}$$

The large N limit is thus singular. During a long lapse of time, the probability distribution (49) presents no narrow peaks in $1/\sqrt{N}$, although it behaves originally as (35) and ends up as a sum of two contributions (20) around $+m_{\mathrm{F}}$ and $-m_{\mathrm{F}}$. Instead, it is a smooth function, independent of N and extending over a wide range of values of m.

Let us illustrate this behavior by describing the evolution of (49) in the completely unbiased case $g = m_0 = \lambda = 0$, for which $P(m,t)$ is symmetric (Fig. 2). The problem is formally similar to the so-called laser model [11] The initial peak (35) with $m_0 = 0$ begins to widen and to reach a finite width. It progressively covers most of the interval $-m_{\mathrm{F}}$, $+m_{\mathrm{F}}$. In particular, at the time

$$t = \theta \ln\left(\frac{m_{\mathrm{F}}}{\delta}\sqrt{\frac{N}{3}}\right) \; , \tag{51}$$

such that $\alpha^2 = \frac{3}{2}$, the distribution

$$P(m,t) = \sqrt{\frac{3}{2\pi}}\frac{m_{\mathrm{F}}^2}{\left(m_{\mathrm{F}}^2 - m^2\right)^{3/2}} \exp\left(-\frac{3}{2}\frac{m^2}{m_{\mathrm{F}} - m^2}\right) \tag{52}$$

is very flat: near $m = 0$, it behaves as $P \propto \exp\left(-3m^4/4m_{\mathrm{F}}^4\right)$; the ratio $P(m,t)/P(0,t)$ is still equal to 0.93 for $m = 0.5m_{\mathrm{F}}$, to 0.84 for $m = 0.6m_{\mathrm{F}}$, to 0.65 for $m = 0.7m_{\mathrm{F}}$. At later times, the origin becomes a minimum and two maxima appear, lying at

$$m = \pm m_{\mathrm{F}}\sqrt{1 - \frac{2\alpha^2}{3}} = \pm m_{\mathrm{F}}\sqrt{1 - \frac{N e^{-2t/\theta} m_{\mathrm{F}}^2}{3\delta^2}} \; . \tag{53}$$

These maxima move away from the origin towards $\pm m_{\mathrm{F}}$. Originally not pronounced, they get narrower and narrower. For sufficiently large times, when α becomes small, they turn into unsymmetrical peaks near m_{F} (and $-m_{\mathrm{F}}$), having the shape

$$P(m)\, dm \sim \frac{e^{-1/x} dx}{2\sqrt{\pi} x^{3/2}} \quad , \quad x = \frac{2}{\alpha^2}\frac{m_{\mathrm{F}} - m}{m_{\mathrm{F}}} \; , \tag{54}$$

with a maximum at $x = \frac{2}{3}$. Both peaks eventually reach the symmetric equilibrium shape (20) when the diffusion term dominates again. We estimate the relaxation time by evaluating after which delay these two peaks arrive at $\pm 0.95 m_{\mathrm{F}}$, which yields

$$\tau_{\mathrm{relax}} = \frac{\hbar}{\gamma(J - T)} \ln\left(\frac{m_{\mathrm{F}}}{\delta}\sqrt{\frac{10N}{3}}\right) \; . \tag{55}$$

This quantity is much larger than the relaxation times which occur in the first regime, by a factor $\ln\sqrt{N}$. The relaxation time becomes even infinite when $N \to \infty$.

For finite values of the bias, as measured by λ defined by (39), (47), the curve (49) is no longer symmetric, but the results are similar. For $\lambda > 0$, the maximum initially at the origin moves as $m \sim \lambda m_{\mathrm{F}}/\alpha$ while widening. At the time (51), there is still a single maximum, but the distribution has become very wide, as for $\lambda = 0$, with larger values for $m > 0$ than for $m < 0$. A second maximum appears later on, at a negative value m_2 of m given by $m_2^3 = -\lambda m_{\mathrm{F}}^2 \sqrt{m_{\mathrm{F}}^2 - 2m_2^2}\,/\sqrt{6}$, and at a time such that $\alpha^2 = \frac{3}{2}\left(m_{\mathrm{F}}^2 - m_2^2\right)\left(m_{\mathrm{F}}^2 - 2m_2^2\right)/m_{\mathrm{F}}^4$. Here again the two maxima end up as narrow peaks near $\pm m_{\mathrm{F}}$, but equilibrium is reached for $+m_{\mathrm{F}}$ earlier than for $-m_{\mathrm{F}}$; the geometric mean of these relaxation times is given by (55). The main difference with the symmetric case is the occurrence of unequal probabilities $\mathscr{P}_+$ and $\mathscr{P}_-$ for the final phases, given by (37), which reads

$$\mathscr{P}_\pm = \frac{1}{2}\operatorname{erfc}(\mp\lambda) \; . \tag{56}$$

These values, determined by the dynamics, are expressed in terms of the various parameters of the problem and of the initial conditions through (39), (47).

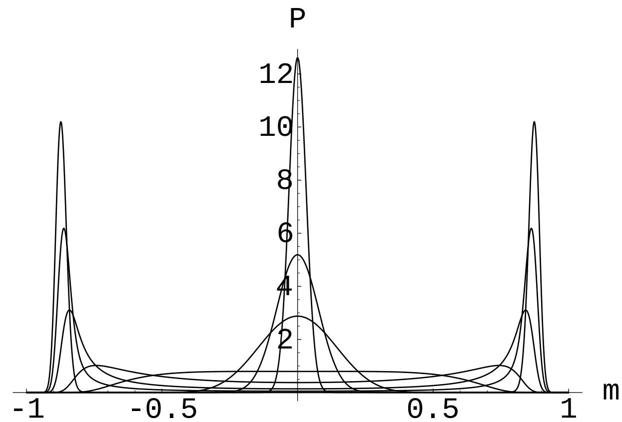

FIGURE 2. Transition from the initial paramagnetic state to the mixture of ferromagnetic states for the same situation as in Fig. 1, $N = 1000$, $T/J = 0.65$ but $g = 0$. The times are: $t/\theta = 0$, 0.5, 1, 2.25, 3, 4, 5 and 10 (from top to bottom at $m = 0$; the last curve is very near the infinite-time limit). The distribution, initially peaked as $1/\sqrt{N}$, becomes flat in Suzuki's scaling regime (49)-(50) (with $\lambda = 0$ here). For large t, two peaks build up near $m_F = 0.87206$ and $-m_F$, with widths governed by the diffusion term $\sim 1/N$.

BURIDAN'S ASS EFFECT

Many among the properties exhibited above have been encountered in different contexts long ago [10,11]. Indeed, the equation (17) which governs the dynamics of the present phase transition has a very general form, describing directed Brownian motion with an unstable fixed point. We have stressed the existence for large N of two contrasted regimes for the behavior of the probability $P(m,t)$, depending on the initial conditions and on the field. We propose to term the second, anomalous situation "Buridan's ass effect". According to an argument attributed to Jean Buridan, a dialectician of the first half of the XIVth century, an ass placed equidistantly from two identical bales of hay will stay there and starve to death because it has no causal reason to choose one or the other [12].

Here, likewise, in a symmetric situation where the initial bifurcation may lead the magnetization m to $+m_\mathrm{F}$ or $-m_\mathrm{F}$ with equal probabilities $\mathscr{P}_+ = \mathscr{P}_- = \frac{1}{2}$, an infinite time (55) is required in the macroscopic limit $N \to \infty$ for the system to relax. We find more generally the same huge duration of the relaxation, from the paramagnetic state to either one of the ferromagnetic states, when there is a bias which is sufficiently small so that neither one of the probabilities $\mathscr{P}_+$ and $\mathscr{P}_-$ for reaching $+m_\mathrm{F}$ and $-m_\mathrm{F}$ vanishes. In such a situation, if N is large but finite, "the ass" m will have reached one or the other bale after the delay (55), and we can predict with which probability: if we perform the same experiment with many asses, we know which proportion will have attained either side. However, for times $t < \theta$ the asses have practically not yet moved, whereas for times such that $e^{t/\theta}$ is of order $\sqrt{N}$, we can make no prediction: the asses are scattered on a wide range between the two bales. This dispersion of the values of m during a long delay is a characteristic feature of the phenomenon.

In contrast, the situation is trivial for a larger bias. This may take place either under the condition (41) when the field g is sufficiently large (a wind which pushes the asses), or under the condition (42) if the initial state has a residual magnetization (the asses are significantly closer to one of the bales). The process then takes a finite time (43) when $N \to \infty$, and we can predict with little error (in $1/\sqrt{N}$) where are the asses at each time.

The giant fluctuations which occur in Buridan's ass effect may be regarded as a dynamical counterpart of those which occur at equilibrium near a critical point. In both cases, the order parameter, although macroscopic, is not a well-defined quantity even in the large N limit, so that its behavior must be described by means of statistical mechanics rather than standard thermodynamics. In the present situation, no temperature can be defined for the system M, during the irreversible process which leads it from a high-temperature equilibrium paramagnetic state (35) to a ferromagnetic equilibrium (20) at the temperature T of the bath, fixed below the critical temperature J. However we may argue that in some sense M crosses its critical point during the dynamical process, in case it has some probability to reach both phases $+m_\mathrm{F}$ and $-m_\mathrm{F}$ in the end. The well-known critical fluctuations and critical slowing down thus manifest themselves here by the large uncertainty about m shown by $P(m,t)$ during the long delay (55). Giant fluctuations in the dynamics are also well-known for nucleation processes. However in the present long-range model (4) there is no space structure, so that we have only to deal with the statistics at each time of the single order parameter m.

CONCLUSION: BURIDAN'S ASS AND QUANTUM MEASUREMENT

As regards the use of $\mathrm{A} = \mathrm{M} + \mathrm{B}$ as an apparatus to measure the spin S, two conditions should be satisfied. On the one hand, the initial metastable paramagnetic state of M, prepared by setting its temperature at $T_0 > J$, should have a long lifetime (55), so that the interaction (3) between S and A can be turned on before M has begun to relax. This is achieved provided there is neither a significant external field g_0 nor a lack of symmetry m_0 before the measurement, a condition expressed more precisely by $\lambda \ll 1$ in (47), that is, using $\delta_0^2 = T_0/(T_0 - J)$,

$$\left(\frac{g_0}{J-T} + m_0\right)^2 \ll \frac{1}{N}\left(\frac{T}{J-T} + \frac{T_0}{T_0 - J}\right) . \tag{57}$$

On the other hand, the coupling g between S and A should be sufficiently large so as to ensure faithfulness of the registration, $+m_{\mathrm{F}}$ in the sector $\uparrow\uparrow$, $-m_{\mathrm{F}}$ in the sector $\downarrow\downarrow$. This is achieved provided Buridan's ass effect does not take place during the measurement process, that is, provided

$$\left(\frac{g}{J-T}\right)^2 \gg \frac{1}{N}\left(\frac{T}{J-T} + \frac{T_0}{T_0 - J}\right) . \tag{58}$$

Then the registration time (43) is finite for large N.

We have chosen to study the model (4) because its exact solution at equilibrium can be obtained for large N through a static mean-field approach; we had the prejudice that its dynamics might also be solved exactly through a time-dependent mean-field approach. It turns out that, although an exact solution in the considered limiting case is available, it cannot reduce to mean-field when Buridan's ass effect is present. A time-dependent mean-field approach relies on the existence of trajectories $m(t)$ around which m has negligible statistical fluctuations. However, when the bias b defined by (47) is small, the dynamical process is governed by fluctuations which become macroscopic. Even if the initial magnetization is exactly defined $(\delta_0 = 0)$ the diffusion term in (17) produces a width (39) which eventually leads to a flat distribution, as illustrated by (52). It is only in the biased regime, which leads the magnetization to a single value $+m_{\mathrm{F}}$ with nearly unit probability, that fluctuations play little role for large N, and that the time-dependent mean-field equation (23) is sufficient to describe the dynamics. When the bifurcation is active, the intuitive idea that the variable m, because it is macroscopic, should display relative fluctuations small as $1/\sqrt{N}$ becomes wrong except near the initial time and near the final equilibrium states.

The two regimes are characterized by the final situation: either non-vanishing probabilities $\mathscr{P}_+$ and $\mathscr{P}_-$ for the two alternatives $+m_{\mathrm{F}}$ and $-m_{\mathrm{F}}$, or a single choice only. Everything takes place, like for Buridan's ass, as if the process were governed by final causes: the behavior is deterministic if the target is unique; it displays large uncertainties in the dynamics if hesitation may lead to one target or the other. In fact, the singularity of the problem arises from the structure of (17), which contains a deterministic drift term involving the velocity $v(m)$ and a diffusion term. Small as $1/N$, the latter term does not contribute significantly when the bias is sufficient to determine a single outcome $(|\lambda| \gg 1)$; it becomes essential when the bifurcation is active (λ finite or small), because it governs the right-hand side of (17) near the fixed points of the drift motion, where $v(m)$ vanishes. The long duration of the relaxation and the large uncertainties in its dynamics then reflect in a probabilistic language the slowness of the pure drift motion around the unstable fixed point, and the long and random delay needed to set m into motion. The very direction of the drift, which will be reflected in the sign of the final magnetization, is thus quite sensitive to the perturbation caused by the diffusion term, which altogether determines the long-time behavior.

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12. The last author has not witnessed such an event.